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On the effect of background voidage on compressive solitary waves in compacting media

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Abstract. The effect of background voidage on the necessary conditions for the existence of compressive solitary wave solutions in the two-phase fluid flow of a medium compacting under gravity is investigated. It is assumed that $K = K_0\phi^n(1-\phi)^{-p}$ and $\xi + \frac{4}{3}\eta = (\xi + \frac{4}{3}\eta)_0\phi^{-m}(1-\phi)^q$, where K is the permeability of the medium, $\xi + \frac{4}{3}\eta$ is the effective viscosity of the solid matrix and ϕ is the voidage. It is shown that for compressive solitary wave solutions to exist, which satisfy certain boundary conditions, it is necessary that the background voidage ϕ_0 and the exponent n lie in two regions of the (ϕ_0, n)-plane when $0 \leqslant p < 1$, and that this reduces to one region when $p \geqslant 1$. Necessary conditions on the exponent m are also derived. Solitary wave solutions for specific values of n, m, p, q and ϕ_0 are obtained numerically and compared.

1. Introduction

Recently, Nakayama and Mason [1] investigated the existence of compressive solitary wave solutions of the third-order nonlinear partial differential equation

$$\frac{\partial\phi}{\partial t} + \frac{\partial}{\partial z}\left[\phi^n\left(1 - \frac{\partial}{\partial z}\left(\frac{1}{\phi^m}\frac{\partial\phi}{\partial t}\right)\right)\right] = 0 \tag{1}$$

derived by Scott and Stevenson [2, 3] and independently for $m = 0$ by Richter and McKenzie [4] and Barcilon and Richter [5] in order to describe the one-dimensional migration of melt through the Earth's mantle. The dependent variable $\phi(z, t)$ is the voidage or volume fraction of melt and $n \geqslant 0$ and $m \geqslant 0$ are constant exponents in power laws relating the permeability of the medium and the bulk and shear viscosities of the solid matrix to the voidage. In a compressive solitary wave a small region of locally low voidage ascends through a background region of higher uniform voidage ϕ_0. In the derivation of (1) it was assumed that ϕ_0 is small and can be approximated by $\phi_0 = 0$. In this paper we will not make the approximation $\phi_0 = 0$ in the governing partial differential equations and we will investigate the effect of the background voidage, ϕ_0, on the existence, speed and shape of the compressive solitary wave. Barcilon and Richter [5] have examined the effect of the background voidage on the speed and shape of rarefactive solitary waves in which a small region of locally high voidage ascends through a background region of lower uniform voidage.

In the study of the migration of melt through the Earth's mantle the values used for ϕ_0 vary from $\phi_0 = 0.01$ to $\phi_0 = 0.05$ [2–7]. Shirley [8] suggests that another probable application of the theory is to the compaction of igneous cumulates in magma chambers.

Igneous cumulates consist of a solid matrix of mineral phases plus interstitial liquid and may be modelled as a mixture of two viscous fluids, namely the melt and the crystal matrix. When initially deposited there may be 50–60% interstitial liquid. Shirley uses an estimate of $\phi_0 = 0.6$ for the melt fraction at the cumulate–magma boundary. He argues that this high estimate for ϕ_0 does not conflict with the results of experiments on the deformation of partially molten granites which indicate a breakdown of the rigid matrix at 0.2–0.35 melt fractions [9–11]. For example, in partially molten tholeiite lava the grains form a rigid interlocking network at melt fractions as high as 0.5 [9] and it is not necessary for the grains to be interlocking to contribute to the stress in the matrix. Jaeger and Cook [12] list the porosity of some porous media which may serve as reference values. For spheres of equal size in a cubic arrangement the porosity is 0.476 while in a closely-packed rhombohedral arrangement it is 0.26. The porosity of lose sand is about 0.4 and for oil sands it is in the range 0.1–0.2.

Essential to the propagation of solitary waves in a medium consisting of melt and a viscous permeable matrix is the dependence of the permeability, K, on the voidage ϕ. We will develop the theory as far as we can without specifying the relationship between K and ϕ and only specify the relationship when it becomes necessary. The law which we will use is

$$K = \frac{K_0 \phi^n}{(1-\phi)^p} \tag{2}$$

where K_0 is a constant and $n \geqslant 0$ and $p \geqslant 0$. When $n = 3$ and $p = 2$, (2) reduces to the Blake–Kozeny–Carman equation which has been shown to agree well with experimental data for $\phi_0 < 0.1$ by McKenzie [13] and for $0.35 < \phi_0 < 0.65$ by Dullien [14]. In the derivation of the partial differential equation (1), equation (2) with $p = 0$ was used. When the voidage is no longer small $p = 0$ may no longer be accurate.

Comparatively little is known about the dependence of the viscosity of the solid matrix on ϕ. The effective viscosity of the solid matrix is [8, 13]

$$\xi + \tfrac{4}{3}\eta = (\xi^* + \tfrac{4}{3}\eta^*)(1-\phi) \tag{3}$$

where ξ^* and η^* are the bulk and shear viscosities of the solid matrix, respectively. We will develop the theory as far as we can without specifying the relationship between the effective viscosity and ϕ. When it becomes necessary to specify a relationship we will use

$$\xi + \tfrac{4}{3}\eta = \frac{(\xi + \frac{4}{3}\eta)_0 (1-\phi)^q}{\phi^m} \tag{4}$$

where $(\xi + \frac{4}{3}\eta)_0$ is a constant and $m \geqslant 0$ and $q \geqslant 0$. Barcilon and Richter [5] assume that ξ and η are constants and therefore that $q = 0$ and $m = 0$. Scott and Stevenson [2] in the derivation of (1) assume that $q = 0$ and $m \geqslant 0$ and suggest that m probably lies in the range 0–1.

An outline of the paper is as follows. The equations which describe two phase fluid flow in a compacting medium are presented in section 2. The approximations for small ϕ_0 are not made. In section 3 travelling wave solutions in the form of a compressive solitary wave are considered and the phase speed of the solitary wave is obtained. In sections 4 and 5 the effect of the permeability of the medium and the viscosity of the solid matrix on the existence of compressive solitary wave solutions is investigated. Details of the analysis are confined to appendices A and B. The effect of the value of n, m and p on the shape of compressive solitary waves is examined in section 6 by considering numerical solutions. Finally, concluding remarks are made in section 7.

2. Basic equations

The theory of two-phase fluid flow in compacting media has been formulated by several authors [2, 3, 13, 15].

We consider a partially molten medium consisting of a solid matrix and a fluid melt which are modelled as two immiscible fully-connected fluids of constant but different densities. The density of the melt is less than the density of the solid matrix. Changes of phase are not included in the model. It is assumed that melting has occurred and only the migration of melt under the action of gravity is considered. (The effect of melting may be included in the governing equations if required [16–18].) The Reynolds numbers for the melt and solid matrix are both much less than unity and therefore inertia effects are neglected. By considering the macroscopic conservation of mass and momentum balance equations for the melt and for the solid matrix, the following two coupled nonlinear partial differential equations for the voidage $\phi(z, t)$ and the z-component of the velocity of the solid matrix $W(z, t)$ may be derived [5, 19]:

$$\frac{\partial \phi}{\partial t} - \frac{\partial}{\partial z}[(1-\phi)W] = 0 \tag{5}$$

$$\frac{\partial}{\partial z}\left[\left(\xi + \frac{4}{3}\eta\right)\frac{\partial W}{\partial z}\right] - \frac{\mu}{K}W - (1-\phi)g\Delta\rho = 0 \tag{6}$$

where the z-coordinate is vertically upwards, μ is the coefficient of shear viscosity of the melt, g is the acceleration due to gravity, $\Delta\rho = \rho_s - \rho_m > 0$ and ρ_s and ρ_m are the densities of the solid matrix and melt, respectively. The barycentric reference frame is used in which

$$\phi w + (1-\phi)W = 0 \tag{7}$$

where w is the z-component of the velocity of the melt.

We introduce dimensionless variables defined by

$$\bar{\phi} = \frac{\phi}{\phi_0} \qquad \overline{W} = \frac{W}{W_0} \qquad \bar{w} = \frac{w}{W_0} \qquad \bar{z} = \frac{z}{\delta_0} \qquad \bar{t} = \frac{t}{t_0} \tag{8}$$

where

$$W_0 = \frac{(1-\phi_0)g\Delta\rho K(\phi_0)}{\mu} \qquad \delta_c = \left(\frac{K(\phi_0)(\xi(\phi_0) + \frac{4}{3}\eta(\phi_0))}{\mu}\right)^{1/2} \tag{9}$$

$$t_0 = \frac{\delta_c \phi_0}{W_0} = \left(\frac{\mu(\xi(\phi_0) + \frac{4}{3}\eta(\phi_0))}{K(\phi_0)}\right)^{1/2} \frac{\phi_0}{(1-\phi_0)g\Delta\rho}. \tag{10}$$

The characteristic length δ_c is the compaction length. We also define

$$k(\bar{\phi}) = \frac{K(\phi_0\bar{\phi})}{K(\phi_0)} \qquad h(\bar{\phi}) = \frac{\xi(\phi_0\bar{\phi}) + \frac{4}{3}\eta(\phi_0\bar{\phi})}{\xi(\phi_0) + \frac{4}{3}\eta(\phi_0)}. \tag{11}$$

We suppress the overhead bars to keep the notation simple. Expressed in dimensionless variables, equations (5) and (6) become

$$\frac{\partial \phi}{\partial t} - \frac{\partial}{\partial z}[(1-\phi_0\phi)W] = 0 \tag{12}$$

$$\frac{\partial}{\partial z}\left(h(\phi)\frac{\partial W}{\partial z}\right) - \frac{W}{k(\phi)} - \frac{(1-\phi_0\phi)}{(1-\phi_0)} = 0. \tag{13}$$

The condition for a barycentric reference frame (7) takes the form

$$\phi_0\phi w + (1-\phi_0\phi)W = 0. \tag{14}$$

Equations (12) and (13) admit the solution $\phi = 1$, $W = -1$, which describes a uniform compaction of solid matrix relative to the melt [5]. This solution is the background state on which the solitary waves propagate. Equation (1) follows directly from (12) and (13) if we make the approximation $\phi_0 = 0$, eliminate W and use (2) with $p = 0$ and (4) with $q = 0$.

Equations (12) and (13) form the basis of the subsequent analysis.

3. Speed of solitary wave

Consider one-dimensional travelling wave solutions of (12) and (13) of the form

$$\phi(z,t) = \psi(\zeta) \qquad W(z,t) = \Omega(\zeta) \qquad \zeta = z - ct \tag{15}$$

where c is the dimensionless speed of the travelling wave. If (15) is substituted into (12) and (13) then the following two ordinary differential equations for ψ and Ω are obtained:

$$c\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} + \frac{\mathrm{d}}{\mathrm{d}\zeta}[(1-\phi_0\psi)\Omega] = 0 \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(h(\psi)\frac{\mathrm{d}\Omega}{\mathrm{d}\zeta}\right) - \frac{\Omega}{k(\psi)} - \frac{(1-\phi_0\psi)}{(1-\phi_0)} = 0. \tag{17}$$

It follows directly from (16) that

$$\Omega = -\frac{(A+c\psi)}{(1-\phi_0\psi)} \tag{18}$$

where A is a constant of integration. If Ω is eliminated from (17) using (18) then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{h(\psi)}{(1-\phi_0\psi)^2}\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right) = \frac{(1-\phi_0)(A+c\psi) - (1-\phi_0\psi)^2 k(\psi)}{(c+\phi_0 A)(1-\phi_0)k(\psi)(1-\phi_0\psi)} \tag{19}$$

provided $c + \phi_0 A \neq 0$. We will see later that $c + \phi_0 A \neq 0$ provided the speed of the solitary wave is not equal to the speed of the compacting solid matrix. Now, for any function $g(\psi)$,

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(g(\psi)\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right) = \frac{1}{2g(\psi)}\frac{\mathrm{d}}{\mathrm{d}\psi}\left(g^2(\psi)\left(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)^2\right) \tag{20}$$

and with the aid of (20), (19) may be integrated once with respect to ψ to give

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)^2 = f(\psi) \tag{21}$$

where

$$f(\psi) = \frac{2(1-\phi_0\psi)^4}{(c+\phi_0 A)(1-\phi_0)h^2(\psi)}\left[B - \int^{\psi} \frac{h(x)[(1-\phi_0 x)^2 k(x) - (1-\phi_0)(A+cx)]\,\mathrm{d}x}{k(x)(1-\phi_0 x)^3}\right] \tag{22}$$

and B is a constant of integration.

In order to obtain the constants A, B and c in (22) we impose the following three boundary conditions at the background state $\psi = 1$ [1, 20–22]:

$$f(1) = 0 \qquad \frac{\mathrm{d}f}{\mathrm{d}\psi}(1) = 0 \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}\psi^2}(1) = 0. \tag{23}$$

The boundary conditions (23) lead to compressive solitary wave solutions in the approximation $\phi_0 = 0$ [1, 20]. If we assume that $k'(1)$, $h'(1)$ and $h''(1)$ are finite, where the prime denotes differentiation with respect to ψ, then we find that

$$f(1) = 0: \qquad B = \int^1 \frac{h(x)[(1-\phi_0 x)^2 k(x) - (1-\phi_0)(A + cx)]\,\mathrm{d}x}{k(x)(1-\phi_0 x)^3} \tag{24}$$

$$f'(1) = 0: \qquad A + c - (1 - \phi_0) = 0 \tag{25}$$

$$f''(1) = 0: \qquad c = (1-\phi_0)k'(1) - 2\phi_0. \tag{26}$$

By solving (24) to (26) for A, B and c and substituting into (22) we obtain

$$f(\psi) = \frac{2(1-\phi_0\psi)^4}{(1-\phi_0)^2[(1-\phi_0)k'(1) - \phi_0]h^2(\psi)} \int_\psi^1 \frac{h(x)G(x)\,\mathrm{d}x}{k(x)(1-\phi_0 x)^3} \tag{27}$$

where

$$G(x) = (1-\phi_0 x)^2 k(x) - (1-\phi_0)[(1-\phi_0)k'(1) - 2\phi_0]x + (1-\phi_0)[(1-\phi_0)k'(1) - (1+\phi_0)]. \tag{28}$$

The dimensionless wave speed c is given by (26). Since the characteristic speed is δ_c/t_0, the phase speed written in dimensional form is

$$c = [(1-\phi_0)k'(1) - 2\phi_0]\frac{\delta_c}{t_0} = [(1-\phi_0)k'(1) - 2\phi_0]\frac{(1-\phi_0)K(\phi_0)g\Delta\rho}{\phi_0\mu}. \tag{29}$$

The phase speed is independent of the bulk and shear viscosities of the solid matrix although it depends on the shear viscosity of the fluid melt μ. It depends significantly on the permeability of the background state through $K(\phi_0)$ and $k'(1)$.

The velocity of the solid matrix in the background state is $W = -1$ and from (14) the velocity of the fluid melt in the background state is $w = (1-\phi_0)/\phi_0$. Expressed in dimensional form we have

$$W = -\phi_0\frac{\delta_c}{t_0} = -(1-\phi_0)K(\phi_0)\frac{g\Delta\rho}{\mu} \tag{30}$$

$$w = (1-\phi_0)\frac{\delta_c}{t_0} = (1-\phi_0)^2 K(\phi_0)g\Delta\rho. \tag{31}$$

We will compare the phase velocity c with W and w in section 4.

For future reference we list results for media in which (2) and (4) are satisfied. Equation (11) becomes

$$k(\phi) = \frac{(1-\phi_0)^p\phi^n}{(1-\phi_0\phi)^p} \qquad h(\phi) = \frac{(1-\phi_0\phi)^q}{(1-\phi_0)^q\phi^m} \tag{32}$$

and (27) takes the form

$$f(\psi) = \frac{2(1-\phi_0\psi)^{4-2q}\psi^{2m}}{(1-\phi_0)^{p-q+2}[n-(n+1-p)\phi_0]}\int_\psi^1 \frac{G(x)\,\mathrm{d}x}{x^{n+m}(1-\phi_0 x)^{3-p-q}} \tag{33}$$

where

$$G(x) = (1-\phi_0)^p(1-\phi_0 x)^{2-p}x^n - (1-\phi_0)[n-(n+2-p)\phi_0]x + (1-\phi_0)[n-1-(n+1-p)\phi_0]. \tag{34}$$

Equation (29) for the phase speed becomes

$$c = [n-(n+2-p)\phi_0]\frac{\delta_c}{t_0} = [n-(n+2-p)\phi_0]\frac{\phi_0^{n-1}K_0 g\Delta\rho}{(1-\phi_0)^{p-1}\mu}. \tag{35}$$

Table 1. Necessary regions of existence in the (ϕ_0, n)-plane for compressive solitary wave solutions satisfying boundary conditions (23). N_1, N_2, N_3, N_4 and N_5 are defined by (41), (45), (46), (42) and (43). On the boundary curves $\phi_0 = N_1(n, p)$ and $\phi_0 = N_5(n, p)$, $f'''(1) = 0$ and higher derivatives of $f(\psi)$ at $\psi = 1$ have to be examined. Solutions do not exist for $\phi_0 = 0$ if $n = 0$ or $n = 1$ for all $p \geqslant 0$.

p	Necessary regions of existence in the (ϕ_0, n)-plane ($n \geqslant 0, 0 \leqslant \phi_0 < 1$)		Range of ψ in compressive solitary wave solution
$0 \leqslant p < 1$	$0 \leqslant \phi_0 \leqslant N_1(n, p)$	(region 1)	Extends to $\psi = 0$
	$N_4(n, p) < \phi_0 \leqslant N_5(n, p)$	(region 2)	Extends to $\psi = 0$
$1 \leqslant p < 2$	$0 \leqslant \phi_0 \leqslant N_1(n, p)$	(region 1)	Extends to $\psi = 0$
$p = 2$	$n > 1, 0 \leqslant \phi_0 < 1$	(region 1)	Extends to $\psi = 0$
$p > 2$	$n > 1, \quad 0 \leqslant \phi_0 < 1$	(region 1A)	Extends to $\psi = 0$
	$0 \leqslant n \leqslant 1, \quad \phi_0 \geqslant N_2(n, p)$	(region 1A)	Extends to $\psi = 0$
	$0 \leqslant n \leqslant 1, N_1(n, p) \leqslant \phi_0 < N_2(n, p)$	(region 1B)	May not extend to $\psi = 0$

4. Effect of permeability on existence of compressive solitary waves

For a compressive solitary wave solution to exist which satisfies the boundary conditions (23) it is necessary that [1]

$$\frac{\mathrm{d}^3 f}{\mathrm{d}\psi^3}(1) \leqslant 0. \tag{36}$$

If $f'''(1) = 0$, further investigation, which we will not undertake here, is required to determine if a compressive solitary wave solution actually exists. If $f'''(1) > 0$, a compressive solitary wave solution superimposed on the background state $\psi = 1$ and satisfying the boundary conditions (23) does not exist.

We will assume that $k'(1)$, $k''(1)$, $h'(1)$, $h''(1)$ and $h'''(1)$ are finite. These conditions are satisfied by (32) provided $\phi_0 \neq 1$. The case $\phi_0 = 1$, however, is excluded because there would be no solid matrix present. It follows directly from (27) and (28) that

$$f'''(1) = -\frac{2[(1-\phi_0)^2 k''(1) - 4\phi_0(1-\phi_0)k'(1) + 2\phi_0^2]}{(1-\phi_0)[(1-\phi_0)k'(1) - \phi_0]}. \tag{37}$$

Equation (37) for $f'''(1)$ depends only on ϕ_0 and the permeability of the medium through $k'(1)$ and $k''(1)$. It is independent of $h(\psi)$ and therefore of the effective viscosity of the solid matrix. From (29) and (30),

$$[(1-\phi_0)k'(1) - \phi_0]\frac{\delta_c}{t_0} = c - W \tag{38}$$

and therefore the denominator of (37) is non-zero provided the solitary wave does not descend at the same speed as the solid matrix.

The quantites $k'(1)$ and $k''(1)$ depend on ϕ_0. To proceed further we now suppose that (32) is satisfied with $n \geqslant 0$ and $p \geqslant 0$. Then, for $0 \leqslant \phi_0 < 1$, the necessary condition (36) takes the form

$$\frac{(n+1-p)(n+2-p)\phi_0^2 - 2n(n+1-p)\phi_0 + n(n-1)}{n-(n+1-p)\phi_0} \geqslant 0. \tag{39}$$

Condition (39) is analysed in appendix A and the results are summarized in table 1. For $0 \leqslant p < 1$ there are two necessary regions of existence of compressive solitary wave solutions in the (ϕ_0, n)-plane, namely

$$\text{region 1: } 0 \leqslant \phi_0 \leqslant N_1(n,p) \qquad \text{region 2: } N_4(n,p) < \phi_0 \leqslant N_5(n,p) \tag{40}$$

where

$$N_1 = \frac{n}{n+2-p} - \frac{1}{(n+2-p)}\left(\frac{n(2-p)}{n+1-p}\right)^{1/2} \tag{41}$$

$$N_4 = \frac{n}{n+1-p} \tag{42}$$

$$N_5 = \frac{n}{n+2-p} + \frac{1}{(n+2-p)}\left(\frac{n(2-p)}{n+1-p}\right)^{1/2}. \tag{43}$$

N_1 and N_5 are the roots of the quadratic form on the numerator of (39). The way in which the necessary regions of existence evolve in the (ϕ_0, n)-plane as p increases from zero is shown in figure 1. For $p \geqslant 1$ there is only one necessary region of existence, namely region 1.

The physical significance of the boundaries $\phi_0 = N_1(n,p)$ and $\phi_0 = N_5(n,p)$ can be established by considering the phase speed in dimensional form. Suppose first that $0 \leqslant p < 1$. It follows directly from (35) that

$$\frac{\partial c}{\partial \phi_0} = (n+2-p)(n+1-p)(N_1-\phi_0)(N_5-\phi_0)\frac{\phi_0^{n-2}K_0 g\Delta\rho}{(1-\phi_0)^p \mu}. \tag{44}$$

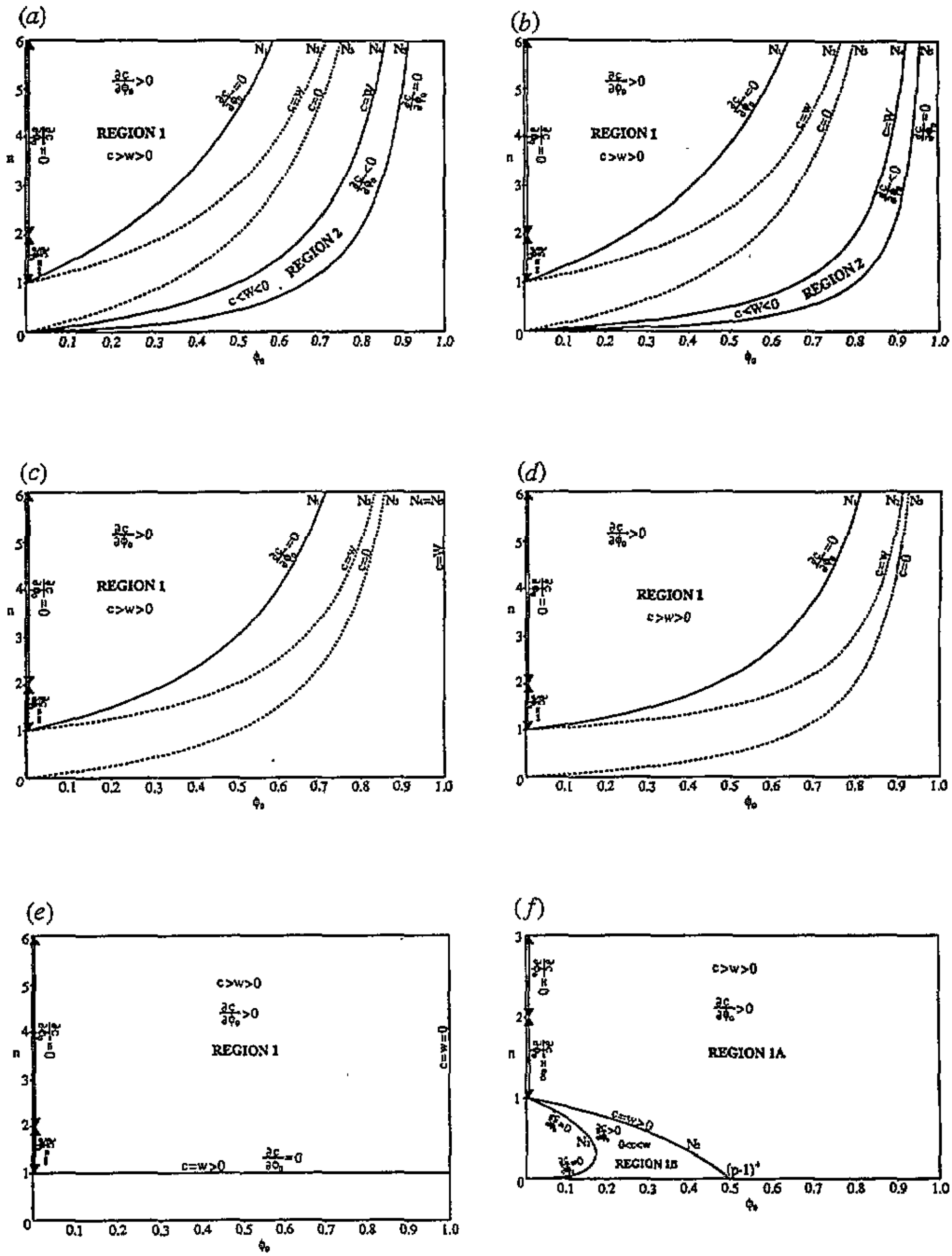

Figure 1. Necessary regions in the (ϕ_0, n)-plane for existence of compressive solitary wave solutions: (a) $p = 0$; (b) $0 < p < 1$ $(p = \frac{1}{2})$; (c) $p = 1$; (d) $1 < p < 2$ $(p = \frac{3}{2})$; (e) $p = 2$; and (f) $p > 2$ $(p = 3)$. The quantities N_1, N_2, N_3, N_4 and N_5 are defined by (41), (45), (46), (42) and (43), c is the velocity of the solitary wave and w and W are the velocities of the fluid melt and solid matrix in the background state.

It also follows from (30), (31) and (35) for W, w and c in dimensional form that $c > w$ if $\phi_0 < N_2(n, p)$ where

$$N_2 = \frac{n-1}{n+1-p} \tag{45}$$

that $c > 0$ if $\phi_0 < N_3(n, p)$ where

$$N_3 = \frac{n}{n+2-p} \tag{46}$$

and that $c > W$ if $\phi_0 < N_4(n, p)$ where N_4 is defined by (42). In the interior of region 1, $c > w > 0$ and $\frac{\partial c}{\partial \phi_0} > 0$; the solitary wave ascends with a speed greater than that of the background melt and c is in increasing function of ϕ_0. On the boundary curve $\phi_0 = N_1(n, p)$ of region 1, $\frac{\partial c}{\partial \phi_0} = 0$. In the interior of region 2, $c < W < 0$ and $\frac{\partial c}{\partial \phi_0} < 0$; the solitary wave descends with a speed greater than the descent speed of the compacting solid matrix and c is a decreasing function of ϕ_0. On the boundary curve $\phi_0 = N_5(n, p)$ of region 2, $\frac{\partial c}{\partial \phi_0} = 0$. For $p \geqslant 1$ the relative values of the velocities c, w and W can be analysed similarly. The results for the relative values of c, w and W for $p \geqslant 0$ are summarized in figure 1.

The range of ψ in a compressive solitary wave solution is investigated in appendix B. When $p > 2$ it is convenient to separate region 1 into two parts as shown in figure 1(*f*):

$$\text{region 1A:} \qquad 0 \leqslant n \leqslant 1 \qquad \phi_0 \geqslant N_2(n, p) \text{ and } n > 1 \qquad 0 \leqslant \phi_0 < 1 \tag{47}$$

$$\text{region 1B:} \qquad 0 \leqslant n \leqslant 1 \qquad N_1(n, p) \leqslant \phi_0 \leqslant N_2(n, p). \tag{48}$$

When $0 \leqslant p \leqslant 2$ and when $p > 2$ and (ϕ_0, n) belongs to region 1A, the range of ψ extends to $\psi = 0$. When $p > 2$ and (ϕ_0, n) belongs to region 1B the range of ψ may not extend to $\psi = 0$ but instead may terminate at $\psi = \psi_{\min} > 0$. The results are summarized in table 1.

5. Effect of matrix viscosity on existence of compressive solitary waves

A solitary wave solution may be identified by the behaviour of $f(\psi)$ near its zeros [23]. A simple zero will correspond to a crest or a trough while a double zero or a triple zero will give an asymptotic tail to ψ near the background state. A compressive solitary wave solution will therefore correspond to a positive solution $f(\psi)$ between the triple zero of $f(\psi)$ at the background state $\psi = 1$ and a *simple* zero at a trough $\psi = \psi_{\min} \geqslant 0$. Thus for a compressive solitary wave solution it is necessary that

$$f(\psi) = (\psi - \psi_{\min})F(\psi) \tag{49}$$

where $0 < F(\psi) < \infty$ for $0 \leqslant \psi_{\min} \leqslant \psi < 1$.

5.1. $0 \leqslant p \leqslant 2$ and $p > 2$ (region 1A)

It was shown in appendix B that the compressive solitary wave solution extends to $\psi = 0$ and therefore $\psi_{\min} = 0$ in (49). The behaviour of $f(\psi)$ as $\psi \to 0$ is summarized in table 2.

Consider first the general case in which (ϕ_0, n) does not lie on the curve $\phi_0 = N_2(n, p)$ and $n \neq 0$. When $n + m > 1$, $f(\psi)$ has a simple zero at $\psi = 0$ provided $m = n$. When $n + m = 1$, $f(\psi)$ does not have a simple zero at $\psi = 0$ and when $0 < n + m < 1$, $f(\psi)$ has a simple zero at $\psi = 0$ provided $m = \frac{1}{2}$. The results for $n + m > 1$ generalize those of Nakayama and Mason [1] for the idealized limit $\phi_0 = 0$.

Consider next the special case in which either $n = 0$ or (ϕ_0, n) lies on the curve $\phi_0 = N_2(n, p)$. Now when $n = 0$ and $\phi_0 = 0$, $G(x) \equiv 0$ from (34) and a compressive solitary wave solution does not exist. We, therefore, see from figure 1 that this special case applies only for $p > 2$. When $m \geqslant 1$, $f(\psi)$ does not have a simple zero at $\psi = 0$ and when $0 \leqslant m < 1$, $f(\psi)$ has a simple zero at $\psi = 0$ provided $m = \frac{1}{2}$.

Table 2. The behaviour of $f(\psi)$ as $\psi \to 0$ for $0 \leqslant p \leqslant 2$ and $p > 2$ (region 1A).

Conditions on n and m	$f(\psi)$ as $\psi \to 0$	Condition for existence of a simple zero of $f(\psi)$ at $\psi = 0$
(a) $n \neq 0$ and $\phi_0 \neq N_2(n, p)$		
(i)		
$n + m > 1$		
$n + m \neq 2$, $m \neq 1$	$\dfrac{2(N_2 - \phi_0)\psi^{m-n+1}}{(N_3 - \phi_0)(n+m-1)(1-\phi_0)^{p-q+1}} \times (1 + \mathrm{O}(\psi^{n+m-1}) + \mathrm{O}(\psi^n) + \mathrm{O}(\psi))$	$m = n$
$n + m = 2$	$\dfrac{2(N_2 - \phi_0)\psi^{m-n+1}}{(N_3 - \phi_0)(n+m-1)(1-\phi_0)^{p-q+1}} \times (1 + \mathrm{O}(\psi^n) + \mathrm{O}(\psi \ln \psi))$	$m = n$
$m = 1$	$\dfrac{2(N_2 - \phi_0)\psi^{m-n+1}}{(N_3 - \phi_0)(n+m-1)(1-\phi_0)^{p-q+1}} \times (1 + \mathrm{O}(\psi) + \mathrm{O}(\psi^n \ln \psi))$	$m = n$
(ii)		
$n + m = 1$	$\dfrac{2(N_2 - \phi_0)\psi^{2m}(-\ln \psi)}{(N_3 - \phi_0)(1-\phi_0)^{p-q+1}} \left(1 + \mathrm{O}\left(\dfrac{\psi^{1-m}}{\ln \psi}\right)\right)$	Simple zero does not exist
(iii)		
$0 < n + m < 1$	$\dfrac{2(N_2 - \phi_0)\psi^{2m}}{(N_3 - \phi_0)(1-n-m)(1-\phi_0)^{p-q+1}} \times (1 + \mathrm{O}(\psi^{1-n-m}))$	$m = \frac{1}{2}$
(b) $n = 0$ or $\phi_0 = N_2(n, p)$		
$m > 1$	$\mathrm{O}(\psi^{m+1})$	Simple zero does not exist
$m = 1$	$\mathrm{O}(\psi^{2m}(-\ln \psi))$	Simple zero does not exist
$0 \leqslant m < 1$	$\mathrm{O}(\psi^{2m})$	$m = \frac{1}{2}$

5.2. *$p > 2$ (region 1B)*

We observed in appendix B that the compressive solitary wave solution may not extend to $\psi = 0$ and we may have $f(\psi_{\min}) = 0$ for some $0 < \psi_{\min} < 1$. For a compressive solitary wave solution to exist it is necessary that $\psi_{\min}$ be a simple zero as stated in (49).

6. Numerical solution for specific values of n, m, p and q

In this section we will compare briefly numerical solutions for specific values of n, m, p and q paying particular attention to the effect of varying p. The exponent q did not occur in the existence criteria and therefore we will always set $q = 0$. Throughout this section we will use as characteristic length

$$\delta_c^* = \left(\frac{K_0(\xi + \frac{4}{3}\eta)_0}{\mu} \right)^{1/2}. \tag{50}$$

Unlike δ_c defined in (9), δ_c^* is independent of n, m, p and q and can be used when comparing solutions with different values of the exponents. The IMSL subroutine DQDAGS, which was designed to integrate functions which have endpoint singularities, will be used in this section when numerical integration is performed [24].

Consider, first, compressive solitary wave solutions with $n = m = 2$ and $p = 0, 1$ and 2. As illustrated in figure 1, for compressive solitary wave solutions to exist it is necessary that ϕ_0 belong to the following ranges: $p = 0$, $0 \leqslant \phi_0 \leqslant 0.106$ and $0.6\dot{6} < \phi_0 \leqslant 0.789$; $p = 1$, $0 \leqslant \phi_0 \leqslant 0.3\dot{3}$; $p = 2$, $0 \leqslant \phi_0 \leqslant 1$. We choose $\zeta = 0$ at $\psi = 0$. The following results may be derived from (21) and (33) with δ_c^* used as characteristic length. We find that $(\frac{d\psi}{d\zeta})^2$ always has a simple zero at $\psi = 0$ consistent with the general theory.

When $n = m = 2$ and $p = 0$,

$$\zeta = \pm\sqrt{3}(1-\phi_0)\int_0^\psi \frac{1}{x^{1/2}(1-\phi_0 x)}\left(\frac{2-3\phi_0}{g(x)}\right)^{1/2} dx \tag{51}$$

where

$$g(x) = (1-x)(A + Bx + Cx^2 + Dx^3 + Ex^4) + Fx^3(1-\phi_0 x)^2 \ln\left(\frac{1-\phi_0 x}{(1-\phi_0)x}\right) \tag{52}$$

and

$$A = 2(1-\phi_0)(1-3\phi_0) \qquad B = (1-\phi_0)(-4 + 11\phi_0 - 15\phi_0^2) \tag{53}$$

$$C = 2 - 9\phi_0 + 66\phi_0^2 - 113\phi_0^3 + 60\phi_0^4 \tag{54}$$

$$D = \phi_0(-4 + 30\phi_0 - 233\phi_0^2 + 456\phi_0^3 - 270\phi_0^4) \tag{55}$$

$$E = \phi_0^2(2 - 17\phi_0 + 144\phi_0^2 - 294\phi_0^3 + 180\phi_0^4) \tag{56}$$

$$F = 6\phi_0(1 - 12\phi_0 + 46\phi_0^2 - 64\phi_0^3 + 30\phi_0^4). \tag{57}$$

When $n = m = 2$ and $p = 1$,

$$\zeta = \pm\sqrt{6}(1-\phi_0)\int_0^\psi \frac{dx}{x^{1/2}(1-\phi_0 x)^{3/2}[g(x)]^{1/2}} \tag{58}$$

where

$$g(x) = (1-x)(A + Bx + Cx^2 + Dx^3) + Ex^3(1-\phi_0 x)\ln\left(\frac{1-\phi_0 x}{(1-\phi_0)x}\right) \tag{59}$$

and

$$A = 2(1-2\phi_0) \qquad B = -4 + 9\phi_0 - 8\phi_0^2 \qquad C = 2 - 9\phi_0 + 31\phi_0^2 - 24\phi_0^3 \tag{60}$$

$$D = \phi_0(-2 + 13\phi_0 - 54\phi_0^2 + 48\phi_0^3) \qquad E = 6\phi_0(1-\phi_0)(1 - 5\phi_0 + 8\phi_0^2). \tag{61}$$

When $n = m = 2$ and $p = 2$,

$$\zeta = \pm\sqrt{3}(2-\phi_0)^{1/2}\int_0^\psi \frac{dx}{x^{1/2}(1-\phi_0 x)^2[g(x)]^{1/2}} \tag{62}$$

where

$$g(x) = (1-x)(A + Bx + Cx^2) + 6\phi_0(1-\phi_0)^2 x^3 \ln\left(\frac{1-\phi_0 x}{(1-\phi_0)x}\right) \tag{63}$$

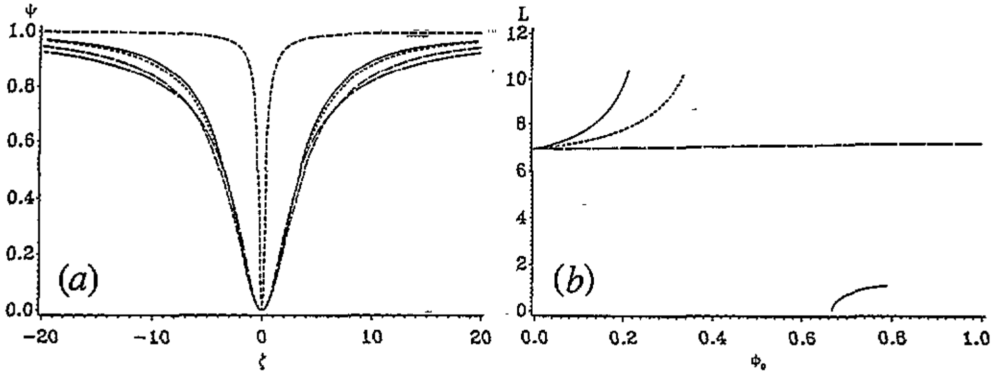

Figure 2. (*a*) Comparison of compressive solitary wave solutions for $n = m = 2$: the analytical solution (65) for all values of p, $\phi_0 = 0$ (——); $p = 0$, $\phi_0 = 0.05$ (----); $p = 0$, $\phi_0 = 0.7$ (– – –); $p = 1$, $\phi_0 = 0.2$ (– – –); and $p = 2$, $\phi_0 = 0.5$ (— — —). (*b*) Width, L, of the compressive solitary wave at half its depth: $p = 0$ (——), $p = 1$ (----) and $p = 2$ (— — —).

and

$$A = 2 \qquad B = -4 + 3\phi_0 \qquad C = 2 - 9\phi_0 + 6\phi_0^2. \tag{64}$$

For $\phi_0 = 0$, the three solutions (51), (58) and (62) reduce to the same solution which can be calculated analytically [1, 20]:

$$\psi = \frac{\zeta^2}{12 + \zeta^2}. \tag{65}$$

For $\phi_0 > 0$ the integrals in (51), (58) and (62) are evaluated numerically. The profiles of the solitary waves for specific values of ϕ_0 are shown in figure 2(*a*).

The width, L, of the compressive solitary wave at half its depth is twice the value of ζ evaluated at $\psi = \frac{1}{2}$. For $\phi_0 = 0$, L is the same for the three solutions and can be calculated analytically [1]:

$$L = 2\sqrt{3} \int_0^{1/2} \frac{\mathrm{d}x}{x^{1/2}(1-x)^{3/2}} = 4\sqrt{3}. \tag{66}$$

For $\phi_0 > 0$, L is evaluated numerically. Graphs of L plotted against ϕ_0 are presented in figure 2(*b*). The width L increases as ϕ_0 increases although, for $p = 0$, L is smaller in region 2 than in region 1.

Consider, next, compressive solitary wave solutions with $n = m = \frac{3}{2}$ and $p = 0, 1$ and 2. For compressive solitary wave solutions to exist it is necessary that ϕ_0 belong to the following ranges as shown in figure 1: $p = 0$, $0 \leqslant \phi_0 \leqslant 0.116$ and $0.6 < \phi_0 \leqslant 0.742$; $p = 1$, $0 \leqslant \phi_0 \leqslant 0.2$; $p = 2$, $0 \leqslant \phi_0 < 1$. We find that $(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta})^2$ has a simple zero at $\psi = 0$ and also that it is a function of $y = \psi^{1/2}$. Negative values of $\psi^{1/2}$ may be included in the range of integration provided $(\frac{\mathrm{d}y}{\mathrm{d}\zeta})^2 > 0$ because the voidage $\psi = y^2 > 0$. We find numerically that $(\frac{\mathrm{d}y}{\mathrm{d}\zeta})^2$ has a negative zero which depends on p and ϕ_0. In the following $y = -\alpha_0$, where $\alpha_0 > 0$, will always denote the negative zero of $(\frac{\mathrm{d}y}{\mathrm{d}\zeta})^2$ with smallest magnitude. The IMSL subroutine DZBREN, which was designed to find a zero of a real function, will be used to obtain the value of $-\alpha_0$ for each case [25]. We find numerically that $y=-\alpha_0$ is a simple zero. We choose $\zeta=0$ at $y=-\alpha_0$ and the range of integration is $-\alpha_0 \leqslant y \leqslant \psi^{1/2}$. The following results may be derived from (21) and (33) with δ_c^* taken as the characteristic length.

When $n=m=\frac{3}{2}$ and $p=0$,

$$\zeta = \pm 2(1-\phi_0)\int_{-\alpha_0}^{\psi^{1/2}} \frac{1}{(1-\phi_0 y^2)}\left(\frac{3-5\phi_0}{g(y)}\right)^{1/2} \mathrm{d}y \tag{67}$$

where

$$\begin{aligned} g(y) &= (1-y)(A+By+Cy^2+Dy^3+Ey^4+Fy^5+Gy^6+Hy^7) \\ &\quad + Ky^4(1-\phi_0 y^2)^2 \ln\left(\frac{1-\phi_0 y^2}{(1-\phi_0)y^2}\right) \\ &\quad + 4\phi_0^{1/2} y^4 (1-\phi_0 y^2)^2 \ln\left(\frac{(1-\phi_0^{1/2}y)(1+\phi_0^{1/2})}{(1+\phi^{1/2}y)(1-\phi_0^{1/2})}\right) \end{aligned} \tag{68}$$

and

$$A = B = (1-\phi_0)(1-5\phi_0) \qquad C = (1-\phi_0)(-5+13\phi_0-20\phi_0^2) \tag{69}$$

$$D = 3+18\phi_0-33\phi_0^2+20\phi_0^3 \qquad E = \phi(10-57\phi_0+131\phi_0^2-90\phi_0^3) \tag{70}$$

$$F = \phi_0(-6-57\phi_0+131\phi_0^2-90\phi_0^3) \qquad G = \phi_0^2(-5+35\phi_0-84\phi_0^2+60\phi_0^3) \tag{71}$$

$$H = \phi_0^2(3+35\phi_0-84\phi_0^2+60\phi_0^3) \qquad K = 6\phi_0(1-\phi_0)(-3+9\phi_0-10\phi_0^2). \tag{72}$$

When $n=m=\frac{3}{2}$ and $p=1$,

$$\zeta = \pm 2\sqrt{3}(1-\phi_0)\int_{-\alpha_0}^{\psi^{1/2}} \frac{\mathrm{d}y}{(1-\phi_0 y^2)^{3/2}[g(y)]^{1/2}} \tag{73}$$

where

$$\begin{aligned} g(y) &= (1-y)(A+By+Cy^2+Dy^3+Ey^4+Fy^5) + Gy^4(1-\phi_0 y^2)\ln\left(\frac{1-\phi_0 y^2}{(1-\phi_0)y^2}\right) \\ &\quad + 4\phi_0^{\frac{1}{2}} y^4(1-\phi_0 y^2)\ln\left(\frac{(1-\phi_0^{1/2}y)(1+\phi_0^{1/2})}{(1+\phi_0^{1/2}y)(1-\phi_0^{1/2})}\right) \end{aligned} \tag{74}$$

and

$$A = B = 1-3\phi_0 \qquad C = -5+10\phi_0-9\phi_0^2 \tag{75}$$

$$D = 3+10\phi_0-9\phi_0^2 \qquad E = \phi_0(5-17\phi_0+18\phi_0^2) \tag{76}$$

$$F = \phi_0(-3-17\phi_0+18\phi_0^2) \qquad G = \phi_0(-12+26\phi_0-18\phi_0^2). \tag{77}$$

When $n=m=\frac{3}{2}$ and $p=2$,

$$\zeta = \pm 2(3-\phi_0)^{1/2}\int_{-\alpha_0}^{\psi^{1/2}} \frac{\mathrm{d}y}{(1-\phi_0 y^2)^2[g(y)]^{1/2}} \tag{78}$$

where

$$g(y) = (1-y)[1+y+(-5+2\phi_0)y^2+(3+2\phi_0)y^3] - 2\phi_0(3-\phi_0)y^4 \ln\left(\frac{1-\phi_0 y^2}{(1-\phi_0)y^2}\right)$$
$$+4\phi_0^{1/2}y^4 \ln\left(\frac{(1-\phi_0^{1/2}y)(1+\phi_0^{1/2})}{(1+\phi_0^{1/2}y)(1-\phi_0^{1/2})}\right). \tag{79}$$

For $\phi_0 = 0$ the three solutions (67), (73) and (78) reduce to the same solution. In each case $-\alpha_0 = -\frac{1}{3}$ and the integral can be evaluated analytically to give [1,20]

$$\psi = \left(\frac{\zeta^2-3}{\zeta^2+9}\right)^2. \tag{80}$$

For $\phi_0 > 0$ the integrals in (67), (73) and (78) are evaluated numerically. The lower limit of integration, $-\alpha_0$, is the negative zero with least magnitude of $(\frac{dy}{d\zeta})^2$ and therefore of $g(y)$. It is found numerically that $0 < |g'(-\alpha_0)| < \infty$ so that $y = -\alpha_0$ is a simple zero of $g(y)$. The values of $-\alpha_0$ and $g'(-\alpha_0)$ for specific values of ϕ_0 and p are listed in table 3. Although $g'(-\alpha_0) < 0$ for $p = 0$ and $0.6 < \phi_0 \leqslant 0.742$, the derivative of $(\frac{dy}{d\zeta})^2$ with respect to y is positive because it contains the factor $(3-5\phi_0)^{-1}$. Graphs of the solitary waves for specific values of p and ϕ_0 are presented in figure 3(a). Each solitary wave has two minima, $\psi = 0$, and one local maximum $\psi = \alpha_0^2$. For $p = 0$ and $0 \leqslant \phi_0 \leqslant 0.116$ the local maximum decreases steadily from $\psi = 0.1\dot{1}$ to $\psi = 0.065$. The solitary wave is always totally compressive. For $p = 0$ and $0.6 < \phi_0 \lesssim 0.742$, the local maximum decreases steadily from $\psi = 1.156$ to $\psi = 0.939$; for $0.6 < \phi_0 \lesssim 0.7$ the local maximum exceeds unity and the solitary wave therefore has a rarefactive part. For $p = 1$ and $0 \leqslant \phi_0 \leqslant 0.2$, the local maximum decreases steadily from $\psi = 0.1\dot{1}$ to $\psi = 0.068$, while for $p = 2$ and $0 \leqslant \phi_0 \leqslant 1$ the local maximum increases slowly from $\psi = 0.1\dot{1}$ to $\psi = 0.134$.

Table 3. The root $-\alpha_0$ of $g(y)$ for specific values of ϕ_0 where $g(y)$ is given by (68), (74) and (79) for $n = m = \frac{3}{2}$ and $p = 0$, 1 and 2, respectively. The magnitude of the local maximum of the solitary wave is $\psi = \alpha_0^2$. Since $0 < |g'(-\alpha_0)| < \infty$, $-\alpha_0$ is a simple zero of $g(y)$.

p	ϕ_0	$-\alpha_0$	α_0^2	$g'(-\alpha_0)$
0, 1, 2	0	$-0.3\dot{3}$	$0.1\dot{1}$	$7.1\dot{1}$
0	0.10	−0.264	0.070	4.124
0	0.61	−1.067	1.138	−6.459
0	0.65	−1.034	1.069	−5.617
0	0.69	−1.003	1.006	−4.967
0	0.70	−0.996	0.992	−4.816
0	0.74	−0.969	0.939	−4.282
1	0.05	−0.320	0.102	6.353
1	0.10	−0.303	0.092	5.545
1	0.15	−0.282	0.080	4.703
1	0.19	−0.261	0.068	3.991
2	0.10	−0.336	0.123	7.071
2	0.50	−0.347	0.120	6.822
2	0.99	−0.366	0.134	6.304

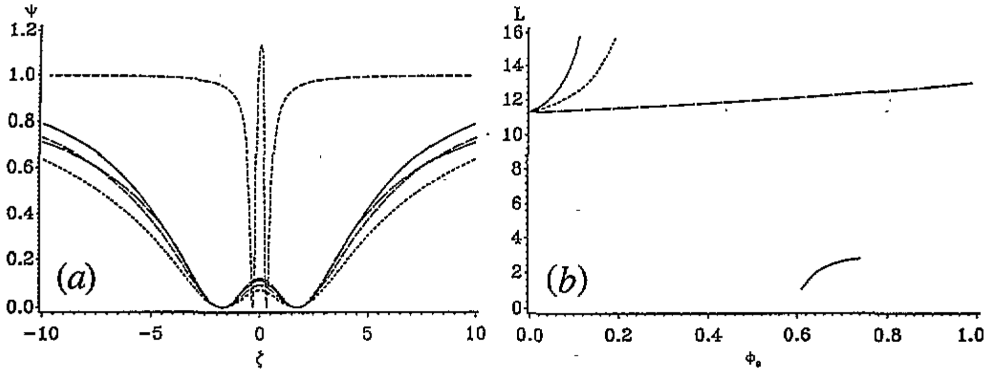

Figure 3. (a) Comparison of compressive solitary wave solutions for $n = m = \frac{3}{2}$: the analytical solution (80) for all values of p, $\phi_0 = 0$ (——); $p = 0$, $\phi_0 = 0.1$ (----); $p = 0$, $\phi_0 = 0.61$ (– – –); $p = 1$, $\phi_0 = 0.1$ (– – –); and $p = 2$, $\phi_0 = 0.5$ (— — —). (b) Total width, L, of the compressive solitary wave at half its depth: $p = 0$ (——), $p = 1$ (----) and $p = 2$ (— — —).

The total width L of the solitary wave at half its depth measured from the background state is twice the value of ζ evaluated at $\zeta = +1/\sqrt{2}$. For $\phi_0 = 0$, L is the same for the three solutions and can be evaluated analytically [1]:

$$L = 4\sqrt{3}\int_{-1/3}^{+1/\sqrt{2}} \frac{\mathrm{d}y}{(1-y)^{3/2}(3y+1)^{1/2}} = 11.31. \tag{81}$$

For $\phi_0 > 0$, L is evaluated numerically and graphs of L plotted against ϕ_0 are presented in figure 3(b). As for the solutions with $n = m = 2$, L increases as ϕ_0 increases although for $p = 0$ the width for ϕ_0 in region 2 is less than in region 1.

The solitary wave solutions for $n = m = 2$ and $n = m = \frac{3}{2}$ ascend, relative to the barycentric reference frame (7), at a speed greater than the ascent speed of the fluid melt in the background state except when ϕ_0 belongs to region 2, in which case the solitary wave descends at a speed greater than the descent speed of the solid matrix in the background state.

Finally, consider compressive solitary wave solutions with $n = m = \frac{1}{2}$ and $p = 3$. We have $N_1 = 0.155$ and $N_2 = 0.33\dot{}$. It follows from the results of section 5 that for $N_2 < \phi_0 < 1$, $f(\psi)$ does not have a simple zero at $\psi = 0$ and therefore a compressive solitary wave solution of the kind considered here does not exist. For $\phi_0 = N_2$, $f(\psi)$ does have a simple zero at $\psi = 0$ since $m = \frac{1}{2}$. Suppose that $N_1 \leqslant \phi_0 \leqslant N_2$. Then from (21) and (33) with δ_c^* taken as the characteristic length,

$$f(\psi) = \left(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)^2 = \frac{2\psi(1-\phi_0\psi)^4}{(1-\phi_0)^4(1+3\phi_0)}g(\psi) \tag{82}$$

where

$$g(\psi) = -(1+\phi_0)(1-\psi) + (1-3\phi_0)\ln\psi + \frac{2(1-\phi_0)^2}{\phi_0^{1/2}}\ln\left(\frac{(1-\phi_0^{1/2}\psi^{1/2})(1+\phi_0^{1/2})}{(1+\phi_0^{1/2}\psi^{1/2})(1-\phi_0^{1/2})}\right). \tag{83}$$

Let $\psi_{\min}$ be the largest zero of $f(\psi)$, and therefore of $g(\psi)$, as ψ decreases from $\psi = 1$. It is found numerically that for $N_1 < \phi_0 < N_2$, $0 < \psi_{\min} < 1$ which is in agreement with the results of section 4. The values of $\psi_{\min}$, which were calculated using subroutine DZBREN [25], with the corresponding values of $f'(\psi_{\min})$ for ϕ_0 in the range $N_1 < \phi_0 \leqslant N_2$ are listed in table 4. Since $0 < f'(\psi_{\min}) < \infty$, it follows that $\psi_{\min}$ is a simple zero of $f(\psi)$. The solitary wave is given by

$$\zeta = \pm \frac{(1-\phi_0)^{1/2}(1+3\phi_0)^{1/3}}{\sqrt{2}} \int_{\psi_{\min}}^{\psi} \frac{d\psi}{\psi^{1/2}(1-\phi_0\psi)^2[g(\psi)]^{1/2}}. \tag{84}$$

Graphs of the solitary wave for specific values of ϕ_0 are presented in figure 4(a). As ϕ_0 increases from N_1 to N_2 the depth of the solitary wave increases steadily from 0 to 1.

Table 4. The zero, $\psi_{\min}$, of $f(\psi)$ where $f(\psi)$ is given by (82) and the corresponding value of $f'(\psi_{\min})$ for $n = m = \frac{1}{2}$, $p = 3$. Since $0 < f'(\psi_{\min}) < \infty$, $\psi_{\min}$ is a simple zero of $f(\psi)$.

ϕ_0	$\psi_{\min}$	$f'(\psi_{\min})$
0.156	0.967	1.369×10^{-6}
0.160	0.871	9.268×10^{-5}
0.180	0.502	9.660×10^{-3}
0.200	0.271	5.045×10^{-2}
0.220	0.133	1.311×10^{-1}
0.240	0.056	2.411×10^{-1}
0.260	0.018	3.522×10^{-1}
0.280	0.003	4.204×10^{-1}
0.300	2×10^{-4}	3.857×10^{-1}
0.320	3×10^{-9}	1.907×10^{-1}
0.330	1×10^{-40}	4.987×10^{-2}

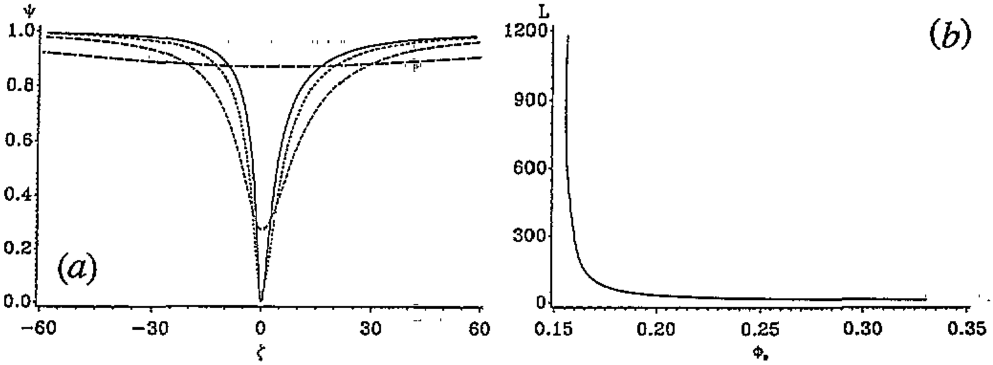

Figure 4. (a) Comparison of compressive solitary wave solutions for $n = m = \frac{1}{2}$, $p = 3$: $\phi_0 = 0.16$ (— — —); $\phi_0 = 0.2$ (– – · –); $\phi_0 = 0.25$ (- - - -); and $\phi_0 = 0.3$ (——). (b) Width, L, of the compressive solitary wave at half its depth for $n = m = \frac{1}{2}$, $p = 3$.

The width, L, of the solitary wave at half its depth is given by twice the value of ζ evaluated at $\psi = \frac{1}{2}(1 + \psi_{\min})$. A graph of L against ϕ_0 is plotted in figure 4(b). As ϕ_0 increases from N_1 to N_2 the width L decreases steadily from infinity to 14.2.

The solitary wave solution for $n = m = \frac{1}{2}$ and $p = 3$ ascends, relative to the barycentric frame (7), at a speed less than the ascent speed of the fluid melt in the background state: $0 < c < w$. This is satisfied by all solitary wave solutions for $p > 2$ with (ϕ_0, n) in region 1B of the (ϕ_0, n)-plane. Further, it is only in this region that the depth of the solitary wave may not extend to $\psi = 0$.

7. Concluding remarks

We have extended the results of Nakayama and Mason [1] on the existence of compressive solitary waves from the idealized case $\phi_0 = 0$ to the complete range of values $0 \leqslant \phi_0 < 1$. The simple power law relating permeability to voidage was generalized to equation (2) which has an empirical basis. There are two regions of existence in the (ϕ_0, n)-plane when $0 \leqslant p < 1$ which evolve into one region for $p \geqslant 1$.

When $m = n > 1$ assumes half-integer values in the idealized limit $\phi_0 = 0$, the solitary waves have oscillatory structure and remain completely compressive [1]. When $n = m = \frac{3}{2}$ and $p = 0$, 1 and 2 the oscillatory solitary waves remain completely compressive for ϕ_0 belonging to region 1 of the (ϕ_0, n)-plane but, for a range of values of ϕ_0 in region 2, part of the oscillatory solitary wave was rarefactive. When considering a fifth order Korteweg–de Vries equation, Kawahara [26] found oscillatory solitary wave solutions which take both rarefactive and compressive values.

A difference between the idealized limit $\phi_0 = 0$ and $\phi_0 > 0$ is that in the latter case compressive solitary wave solutions exist which do not extend to $\psi = 0$. This occurs only when $p > 2$ and (ϕ_0, n) belongs to region 1B of the (ϕ_0, n)-plane. We illustrated this exceptional case by considering the solution for $n = m = \frac{1}{2}$ and $p = 3$. The depth of the solitary wave increased from 0 to 1 as ϕ_0 increased from $N_1 = 0.155$ to $N_2 = 0.33$.

The ascent speed of the solitary wave exceeded the ascent speed of the fluid melt in the background state except in two cases. The first case was for $0 \leqslant p < 1$ when (ϕ_0, n) belongs to region 2. The solitary wave descended with a speed greater than the descent speed of the solid matrix in the background state. The second case was for $p > 2$ when (ϕ_0, n) belongs to region 1B. The solitary wave ascended at a speed less than that of the fluid melt. For this case the depth of the solitary wave may be less than unity. All velocities were measured relative to a barycentric reference frame.

We found that the width of the solitary wave at half its depth increased as ϕ_0 increased for $p = 0, 1, 2$ with $n = m = 2$ and $n = m = \frac{3}{2}$, although the width in region 2 was less than that in region 1. However, for $p = 3$ and $n = m = \frac{1}{2}$, the width decreased as ϕ_0 increased.

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Appendix A. Necessary regions of existence in the (ϕ_0, n)-plane

The discriminant, Δ, of the quadratic form on the numerator of (39) is

$$\Delta = 4n(n+1-p)(2-p). \tag{A1}$$

The subsequent analysis depends on the value of p. We will outline the analysis for $0 \leqslant p < 1$ and then state the results for $p \leqslant 1$ which are derived similarly.

When $0 \leqslant p < 1$ the quadratic form on the numerator of (39) has two real roots, N_1 and N_5, defined by (41) and (43), which are distinct if $n > 0$. The necessary condition (39) takes the form

$$\frac{(N_1 - \phi_0)(N_5 - \phi_0)}{(N_4 - \phi_0)} \geqslant 0 \tag{A2}$$

where N_4 is defined by (42). Graphs of the curves $\phi_0 = N_1(n, p)$, $\phi_0 = N_4(n, p)$ and $\phi_0 = N_5(n, p)$ are illustrated in figures 1(*a*) and 1(*b*) for $p = 0$ and $p = \frac{1}{2}$, respectively. It can be verified that $N_1 < 0$ for $0 < n < 1$, $0 < N_1 < 1$ for $n > 1$, $0 < N_4 < 1$ and $0 < N_5 < 1$ for $n > 0$ and $N_1 < N_4 < N_5$ for $n > 0$. On the curve $\phi_0 = N_4(n, p)$, $f(\psi) = \infty$ by (33) and a compressive solitary wave solution does not exist. For a compressive solitary wave solution to exist it is therefore necessary that the point (ϕ_0, n) lie either in region 1 or region 2 defined by (40) and illustrated in figures 1(*a*) and 1(*b*). On the boundary curves $\phi_0 = N_1(n, p)$ and $\phi_0 = N_5(n, p)$ higher derivatives of $f(\psi)$ at $\psi = 1$ would have to be examined to determine if a compressive wave solution actually exists. A compressive solitary wave solution does not exist at the boundary points $\phi_0 = 0$, $n = 1$ and $\phi_0 = 0$, $n = 0$ for all $p \geqslant 0$, because $G(x) \equiv 0$ by (34).

The evolution of the necessary region of existence as p increases from $p = 1$ to values of $p > 2$ is illustrated in figures 1(*c*) to 1(*f*). When $p = 1$, $N_4 = N_5 = 1$. For $p \geqslant 1$ there is only one necessary region of existence, namely region 1. As p increases from 1 to 2, region 1 extends further into the (ϕ_0, n)-plane. When $p = 2$ and $n = 1$, $G(x) \equiv 0$ and a compressive solitary wave solution does not exist; when $p = 2$, region 1 consists of the part $n > 1$ of the (ϕ_0, n)-plane. When $p > 2$, region 1 extends into the part $n < 1$ of the (ϕ_0, n)-plane. In appendix B, region 1 will be subdivided into regions 1A and 1B as shown in figure 1(*f*).

Appendix B. Range of ψ in a compressive solitary wave solution

We outline the analysis for $0 \leqslant p < 1$ and state the results for $p \geqslant 1$.

Suppose that $0 \leqslant p < 1$. We show that when a compressive solitary wave solution exists the range of ψ extends to $\psi = 0$. From (34), $G(1) = 0$, $G'(1) = 0$ and if $n > 0$,

$$G(0) = (1 - \phi_0)(n + 1 - p)(N_2 - \phi_0). \tag{B1}$$

Also

$$\frac{\mathrm{d}^2 G}{\mathrm{d}x^2} = (n + 1 - p)(n + 2 - p)(N_1 - \phi_0 x)(N_5 - \phi_0 x)\frac{(1 - \phi_0)^p x^{n-2}}{(1 - \phi_0 x)^p}. \tag{B2}$$

Consider first points (ϕ_0, n) which lie in region 1 so that $0 \leqslant \phi_0 \leqslant N_1(n, p)$. Then $G''(x) > 0$ for $0 < x < 1$ and therefore $G(x)$ is concave up for $0 < x < 1$. Since $G(1) = 0$ and $G'(1) = 0$ it follows that $G(x) > 0$ for $0 \leqslant x < 1$. Also the terms outside the integral in (33) are positive since $\phi_0 < N_4(n, p)$. Hence $f(\psi) > 0$ for $0 < \psi < 1$ and the range of the compressive solitary wave extends to $\psi = 0$. Consider next points (ϕ_0, n) which lie in region 2 so that $N_4(n, p) < \phi_0 \leqslant N_5(n, p)$. If $0 < n \leqslant 1$ then $N_1 \leqslant 0$ and therefore $G''(x) < 0$ for $0 < x < 1$; $G(x)$ is concave down for $0 < x < 1$ and since $G(1) = 0$ and $G'(1) = 0$ if follows that $G(x) < 0$ for $0 \leqslant x < 1$. If $n > 1$ then $0 < N_1(n, p) < 1$ and $G''(x) < 0$ for $N_1(n, p)/\phi_0 < x < 1$ and $G''(x) > 0$ for $0 < x < N_1(n, p)/\phi_0$; $G(x)$ is concave down for $N_1(n, p)/\phi_0 < x < 1$ and concave up for $0 < x < N_1(n, p)/\phi_0$. Since $G(1) = 0$, $G'(1) = 0$ and $G(0) < 0$ it follows that $G(x) < 0$ for $0 < x < 1$, but the terms outside the integral in (33) are negative when $\phi_0 > N_4(n, p)$. Thus $f(\psi) > 0$ for $0 < \psi < 1$ and the range of the solitary wave extends to $\psi = 0$.

Similarly it can be shown, when $1 \leqslant p \leqslant 2$ and also when $p > 2$ and (ϕ_0, n) belongs to region 1A defined by (47), that the range of ψ extends to $\psi = 0$. When $p > 2$ and (ϕ_0, n) belongs to region 1B defined by (48), the range of ψ may not extend to $\psi = 0$; since $G(0) < 0$ it follows that $G(x) < 0$ for part of the interval $0 \leqslant x \leqslant 1$ and therefore $f(\psi)$ may vanish at some point $\psi = \psi_{\min} > 0$.

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